Abelian CM Varieties with $\mathbb{Q}(i)$-multiplication of Type $(2,1)$ and Singular Moduli

Thorsten Riedel

Abstract

It is well-known that principally polarized abelian varieties with $\mathbb{Q}(i)$-multiplication of type $(2,1)$ are parametrized by a complex 2-dimensional ball. In this note we show that set of ball points parametrizing CM varieties coincides with the set of cubic $\mathbb{Q}(i)$-singular moduli, i.e. with the set of fixed points of the unitary group $\mathrm{U}((2,1),\mathbb{Q}(i))$ acting on the ball which generate cubic extensions of the Gauss number field.

1 Introduction

To present our results we have to fix some notation. Let $K = \mathbb{Q}(i)$ denote the Gauss number field and let

$$\mathrm{U}(2,1) := \{\gamma \in \mathrm{GL}(3,\mathbb{C}) : {}^t\gamma H \bar{\gamma} = H\},\ H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the unitary group associated to the hermitian form H of signature $(2,1)$. The group $\mathrm{U}(2,1)$ and its subgroups

$$\mathrm{U}((2,1),K) := \mathrm{U}(2,1) \cap M(3,K),\ G := \mathrm{U}(2,1) \cap \mathrm{M}(3,\mathbb{Z}[i])$$

operate canonically on $\mathbb{P}^2_{\mathbb{C}}$ and also on the projective ball

$$\mathbb{B} := \{(a_1 : a_2 : a_5) \in \mathbb{P}^2_{\mathbb{C}} : (a_1,a_2,a_5)H^t(\bar{a}_1,\bar{a}_2,\bar{a}_5) < 0\} \subset \mathbb{P}^2_{\mathbb{C}}.$$

Via $(u,v) \mapsto (1 : u : v) =: \mathbb{P}(u,v)$ we may identify $\mathbb{B}$ with the complex domain

$$\mathbb{B} = \{(u,v) : \mathrm{Im}(u) - \frac{1}{2}|v|^2 > 0\} \subset \mathbb{C}^2.$$

Let A be a principally polarized abelian variety of dimension 3. By the complex representation we consider the endomorphism algebra $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$

Thorsten Riedel, Technische Universität Braunschweig, Institut für Analysis und Algebra, Pockelsstr. 14, 38106 Braunschweig, Germany. E-mail: thorsten_riedel@gmx.de

Mathematics Subject Classification (2010): 11G15, 14K22

Keywords: Complex multiplication of abelian varieties, hyperelliptic curves

as subalgebra of $M(3,\mathbb{C})$. We say that A has K-multiplication of type $(2,1)$ if there is an embedding ι of K into $\mathrm{End}^0(A)$ such that $\iota(f) = \mathrm{diag}(f,f,\bar{f})$ for all $f \in K$. As in the elliptic curve case we define CM varieties as simple abelian varieties with largest possible endomorphism algebra; in our situation this means that $\mathrm{End}^0(A) \cong F$ with a sextic number field F. We denote by $\Gamma = \mathrm{P}G < \mathrm{PGL}(3,\mathbb{C})$ the Picard modular group of the Gauss numbers. It is well-known that the Bailey-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of the ball quotient $\mathbb{B}/\Gamma$ is the moduli space of principally polarized abelian varieties with K-multiplication of type $(2,1)$. A CM point is a ball point that parametrizes a CM variety.

The goal of this note is to give a characterisation of CM points that don't require knowledge about the corresponding variety.

Definition 1.1. *A ball point $\tau \in \mathbb{B}$ is called a* K-singular modulus *if there is a transformation $\gamma \in \mathrm{U}((2,1),K)$ with isolated fixed point τ.*

A result of Feustel [Feu90] applied to our situation tells us that a K-singular modulus generates a field extension of K of degree less or equal 3. We will see that for cubic singular moduli, i.e. for singular moduli τ with $[K(\tau):K] = 3$, this field is isomorphic to the endomorphism algebra of a CM variety, and contrary, that every endomorphism algebra of a CM variety is generated by a singular modulus. From this fact we conclude that the cubic singular moduli are exactly the CM points in $\mathbb{B}$.

Every CM variety of dimension 3 with K-multiplication of type $(2,1)$ appears as Jacobian of a hyperelliptic curve. As an application we obtain that every sextic CM field containing the Gauss numbers arises as endomorphism algebra of a Jacobian of this family of curves.

This paper is based on [Rie11].

2 Singular Moduli

In order to study abelian varieties with imaginary quadratic multiplication, Holzapfel [Hol94] introduced the notion of a Picard matrix.[1] To apply this concept we define an embedding $* : \mathbb{C}^3 \to \mathbb{C}^6$ by

$$^t(z_1, z_2, z_3) \mapsto {}^t(z_1, -iz_1, z_3, z_2, -iz_2, -iz_3).$$

Now the hermitian form $H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $\mathbb{C}^3$ reads as

$$h(z,w) := {}^t(*z)\left(-\frac{i}{2}J\right)(\overline{*w}) = {}^t z H \bar{w} \quad \text{with } J = \begin{pmatrix} 0 & 1_3 \\ -1_3 & 0 \end{pmatrix}.$$

Definition 2.1. *We call*

$$\Pi := {}^t\begin{pmatrix} *a & \overline{*b} & \overline{*c} \end{pmatrix} := \begin{pmatrix} a_1 & -ia_1 & a_5 & a_2 & -ia_2 & -ia_5 \\ \bar{b}_1 & i\bar{b}_1 & \bar{b}_5 & \bar{b}_2 & i\bar{b}_2 & i\bar{b}_5 \\ \bar{c}_1 & i\bar{c}_1 & \bar{c}_5 & \bar{c}_2 & i\bar{c}_2 & i\bar{c}_5 \end{pmatrix} \tag{1}$$

[1] Definition 2.1 below is a slight modification of Holzapfel's notation of a Picard matrix

a Picard matrix *if the following conditions are fulfilled*

(i) $\tau = \mathbb{P}a \in \mathbb{B}$,

(ii) $h(a,b) = h(a,c) = 0$,

(iii) b *and* c *are linearly independent.*

Proposition 2.2. *Let* $A_\Pi = \mathbb{C}^3/\Pi\mathbb{Z}^6$, Π *as in (1), be a principally polarized abelian variety. Then* Π *is a Picard matrix.*

Proof. If $A_\Pi = \mathbb{C}^3/\Pi\mathbb{Z}^6$ is a principally polarized abelian variety, then Π satisfies the Riemann relations

$$\Pi J^t\Pi = 0, \quad i\Pi J^t\bar{\Pi} > 0.$$

Let e_k denote the canonical k-th basis vector. The Riemann relations yield

$$0 > -\frac{1}{2}\,{}^te_1 i\Pi J^t\bar{\Pi}e_1 = {}^t(*a)\frac{-i}{2}J^t(\overline{*a}) = h(a,a)$$

and

$$0 = -\frac{i}{2}{}^te_1\Pi J^t\Pi e_2 = h(a,b), \quad 0 = -\frac{i}{2}{}^te_1\Pi J^t\Pi e_3 = h(a,c).$$

Since Π has rank 3, the vectors b and c are linearly independent thus Π is a Picard matrix. $\square$

Every abelian variety defined by a Picard matrix has multiplication by K, see [Hol94]. To unveil this item we recall some facts from [Hol94], Ch. 6 and 7. In addition we present the key (Theorem 2.3 below) to the interplay between CM points and cubic singular moduli.

For this purpose we have to introduce the following notions, cp. [Hol94], Ch. 7. Let $z = {}^t(z_1, z_2, z_3) \in \mathbb{C}^3$ and $C = {}^t(r_1\ r_2\ r_3) \in M(3,\mathbb{C})$ a matrix with rows r_1, r_2, r_3. We define

$$z^\wedge := {}^t(z_1, \bar{z}_2, \bar{z}_3) \text{ and } C^\wedge := {}^t(r_1\ \bar{r}_2\ \bar{r}_3).$$

If $\Pi := {}^t\begin{pmatrix} *a & \overline{*b} & \overline{*c}\end{pmatrix}$ is a Picard matrix, then

$$\mathbb{Q} \otimes \Pi\mathbb{Z}^6 = \{({}^t(a\ b\ c)z)^\wedge : z \in K^3\}.$$

So every $C \in \mathrm{End}^0(A_\Pi)$ comes along with some $\gamma_C \in M(3,K)$ such that

$$C^t(a\ \bar{b}\ \bar{c}) = ({}^t(a\ b\ c)^t\gamma_C)^\wedge.$$

By $\hat{\imath}(f) := \mathrm{diag}(f, \bar{f}, \bar{f})$ we embed K into $\mathrm{End}^0(A_\Pi)$ and by $\iota(f) := \hat{\imath}(\bar{f})$ we get a $(2,1)$-multiplication on A_Π.

Assume C commutes with every $\hat{\imath}(f)$, $f \in K$. Then $\gamma_C(a\ b\ c) = (a\ b\ c)^tC^\wedge$ and

$$\gamma_C \in \{\gamma \in M(3,K) : \gamma a \in \mathbb{C}a, \gamma a^\perp \subset a^\perp\} =: \mathrm{End}_K(a, a^\perp).$$

Moreover it holds

Theorem 2.3 (Holzapfel, cp. [Hol94], 7.7 Proposition). *The correspondence $C \mapsto \gamma_C$ induces an isomorphism of $\mathbb{Q}$-algebras*

$$Z_{\mathrm{End}^0(A_\tau)}\hat{\iota}(K) \to \mathrm{End}_K(a, a^\perp),$$

where $Z_{\mathrm{End}^0(A_\tau)}\hat{\iota}(K)$ denotes the centralizer of $\hat{\iota}(K)$ in $\mathrm{End}^0(A_\tau)$.

The next lemma tells us how a period quotient of a Picard matrix looks like. Let $\mathbb{H}_3$ denote the Siegel upper halfspace of degree 3 consisting of complex symmetric 3×3-matrices with positive definite imaginary part.

Lemma 2.4. *Suppose that $\tau = \mathbb{P}a = (a_1 : a_2 : a_5) \in \mathbb{B}$ and let $\Pi = {}^t\begin{pmatrix} *a & \overline{*b} & \overline{*c}\end{pmatrix}$ be a Picard matrix. Set*

$$\Omega := \begin{pmatrix} u + \frac{i}{2}v^2 & -\frac{1}{2}v^2 & -iv \\ -\frac{1}{2}v^2 & u - \frac{i}{2}v^2 & v \\ -iv & v & i \end{pmatrix},\ u := \frac{a_2}{a_1},\ v := \frac{a_5}{a_1}. \tag{2}$$

Then Ω lies in the upper halfspace $\mathbb{H}_3$ and

$$A_\Pi = \mathbb{C}^3/\Pi\mathbb{Z}^6 \cong \mathbb{C}^3/(1_3|\Omega)\mathbb{Z}^6 =: A_\tau.$$

Especially Ω and the isomorphism class of A_Π depends only on the chosen ball point.

Proof. Obviously $\Omega = {}^t\Omega$. We write Π as $\Pi = (\Pi_1|\Pi_2)$ with $\Pi_k \in M(3, \mathbb{C})$. Direct calculations show that $\mathrm{Im}(\Omega) > 0$ if $\tau \in \mathbb{B}$ and that $\Pi_1\Omega = \Pi_2$ if $h(a, b) = 0 = h(a, c)$. □

In the sequel we do not distinguish between A_Π and the isomorphic variety A_τ parametrized by a ball point.

Lemma 2.5. *Assume that $\tau = \mathbb{P}a$ is a K-singular modulus, i.e. there is some $\gamma \in \mathrm{U}((2, 1), K)$ with isolated fixed point τ. Let λ denote the corresponding eigenvalue. Then $K(\lambda) = K(\tau) := K\left(\frac{a_2}{a_1}, \frac{a_5}{a_1}\right)$.*

Proof. Without loss of generality we suppose $a = {}^t(1, u, v)$. Since $\gamma a = \lambda a$, it holds $g_{11} + g_{12}u + g_{13}v = \lambda$, where $\gamma = (g_{ij})_{1 \le i,j \le 3}$, so $K(\lambda) \subset K(\tau)$.
Conversely, consider $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma|_{K(\lambda)} = \mathrm{id}$. Then $\gamma(a^\sigma) = (\gamma a)^\sigma = \lambda a^\sigma$. Thus a and a^σ are eigenvectors of the same simple eigenvalue hence they are linearly dependent. Because of their first coordinate, it is $a^\sigma = a$. So σ fixes u and v thus $K(\tau) \subset K(\lambda)$. □

Theorem 2.6 (Feustel, see [Feu90]). *Let $\tau = \mathbb{P}a \in \mathbb{B}$ and let $F = K(\tau)$ be projectively generated by τ. Let N denote the normal closure of F/K. Then τ is a K-singular modulus if and only if the following conditions are satisfied*

(i) $[F : K] \le 3$,

(ii) *F is a CM field,*

(iii) *for all $\sigma \in G(N/K)$ with $\sigma\,|_F \neq \mathrm{id}_F$ it holds $h(a, a^\sigma) = 0$.*

In particular, we are interested in cubic singular moduli, i.e. in K-singular moduli with $[K(\tau) : K] = 3$.

Proposition 2.7. *Let $\tau = \mathbb{P}a = (1 : u : v) \in \mathbb{B}$ and let N be the normal closure of $K(\tau)/K$. Assume that $h(a, a^\sigma) = 0$ for all $\sigma \in G(N/K)$ with $\sigma|_{K(\tau)} \neq \mathrm{id}_{K(\tau)}$.*

(i) *If A_τ is a CM variety, we have $\mathrm{End}^0(A_\tau) \cong K(\tau)$.*

(ii) *Conversely, let $K(\tau)$ be a sextic CM field. Then A_τ is a CM variety.*

Proof. (i) The endomorphism algebra of a CM variety A_τ is a cubic field extension of K. Theorem 2.3 yields

$$\mathrm{End}^0(A_\tau) = Z_{\mathrm{End}^0(A_\tau)}\hat{\iota}(K) \cong \mathrm{End}_K(a, a^\perp).$$

Let $K(\alpha) \cong \mathrm{End}^0(A_\tau)$ and denote by γ_α the image of α in $\mathrm{End}_K(a, a^\perp)$. The minimal polynomial of γ_α in $K[t]$ is irreducible of degree 3, so it has three different roots. Thus a is eigenvector of a simple eigenvalue λ. With Lemma 2.5 it follows

$$K(\tau) \cong K(\lambda) \cong K(\gamma_\alpha) \cong \mathrm{End}^0(A_\tau).$$

(ii) Theorem 2.6 says that τ is a cubic singular modulus, thus τ is fixed by an element $\gamma \in \mathrm{U}((2,1), K)$. Once more Lemma 2.5 and Holzapfel's Theorem 2.3 yield

$$K(\tau) \cong K(\gamma) \hookrightarrow \mathrm{End}^0(A_\tau).$$

Hence the variety is isogenous to a product of CM varieties. The 3 different embeddings of the sextic CM field $K(\tau)$ into $\mathbb{C}$ induced by $K(\tau) \hookrightarrow \mathrm{End}^0(A_\tau)$ define a primitive CM type on $K(\tau)$. It follows: A_τ is simple and therefore a CM variety. $\square$

It is not difficult to see that a given sextic CM field F containing the fourth roots of unity is generated by some $\tau \in \mathbb{B}$. It remains to answer the question if it is always possible to satisfy the additional condition on τ.

Theorem 2.8. *The following assertions are equivalent:*

(i) *A_τ is a CM variety,*

(ii) *τ is a cubic K-singular modulus.*

Proof. Theorem 2.3 states that A_τ is a CM variety if and only if $\mathrm{End}^0(A_\tau) \cong \mathrm{End}_K(a, a^\perp) =: F$ is a cubic extension of K.

$((i) \Rightarrow (ii))$ Choose a primitive element $\gamma \in \mathrm{End}_K(a, a^\perp) = F$ of F/K and an eigenvalue λ with eigenvector a. As in the proof of Proposition 2.7 we deduce

$$F \cong K(\gamma) \cong K(\lambda).$$

Let $\mathrm{id}, \sigma, \psi : F \to \mathbb{C}$ be the different K-embeddings of F. Then

$$\gamma a^\sigma = (\gamma a)^\sigma = (\lambda a)^\sigma = \lambda^\sigma a^\sigma \text{ and } \gamma a^\psi = (\gamma a)^\psi = (\lambda a)^\psi = \lambda^\psi a^\psi,$$

i.e. a, a^σ and a^ψ are eigenvectors of λ, λ^σ and λ^ψ. Since $\gamma(a^\perp) \subset a^\perp$, the eigenvectors a^σ, a^ψ are in the complement $a^\perp$. We get $F = K(\tau)$ by Lemma 2.5, so Theorem 2.6 asserts that τ is a cubic K-singular modulus.

($(ii) \Rightarrow (i)$) Assume that τ is fixed by $\gamma \in \mathrm{U}((2,1), K)$ with corresponding eigenvalue λ. Set $\mathrm{Hom}_K(K(\tau), \mathbb{C}) = \{\mathrm{id}, \sigma, \psi\}$. By Theorem 2.6 the complement $a^\perp$ is generated by a^σ and a^ψ. It follows immediately $\gamma(a^\perp) = a^\perp$, therefore $\gamma \in \mathrm{End}_K(a, a^\perp)$. Because of $K(\tau) \cong K(\lambda) \cong K(\gamma) \subset \mathrm{End}^0(A_\tau)$, the degree $[K(\tau) : K] = 3$ implies that A_τ is a CM variety. □

Finally we want to mention that the abelian varieties considerered above are Jacobian varieties. Consider the family $\mathcal{F}$ of hyperelliptic curves $\widetilde{C}(x, y)$ of genus 3 with model

$$C(x, y) : w^4 = z^2(z-1)^2(z-x)(z-y)$$

and parameters $(x, y) \in \{(x, y) : xy(x-1)(y-1)(x-y) = 0\} \subset \mathbb{C}^2$.

By Matsumoto's paper [Mat89], we know that every isomorphism class of Jacobians is represented by an abelian variety with underlying complex torus $\mathbb{C}^3/\Pi\mathbb{Z}^6$ with $\Pi = {}^t\begin{pmatrix} *a & \overline{*b} & \overline{*c}\end{pmatrix}$. Because Π has to be a Picard matrix (Prop. 2.2) we know also an associated period quotient Ω (see (2) above). This result was first obtained by Matsumoto calculating relations between period integrals.

For a given sextic CM field F containing K, there is a principally polarized CM variety with F-multiplication of type $(2, 1)$ (restricted to K). Because every simple abelian varity A_τ is Jacobian of a curve $\widetilde{C}(x, y) \in \mathcal{F}$, we obtain

Theorem 2.9. *Every sextic CM field containing $\mathbb{Q}(i)$ occurs as endomorphism algebra of a Jacobian variety of a curve of the family $\mathcal{F}$.*

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